Seat No. : _______________

ZQ-118

May-2014

Sem.-II

409 : Mathematics

(Complex Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Suppose f is analytic on an annular domain $R_1 < |z - z_0| < R_2$, and C is any positively oriented simple closed contour around z_0 and lying in this annular domain. Show that at each point z in the domain, f(z) has the series representation **7**

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad R_1 < | z - z_0| < R_2$$

How are the coefficients a_n and b_n given ?

OR

Suppose z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. If $R_1 = |z_1 - z_0|$, show that this power series converges uniformly in the closed disk $|z - z_0| \leq R_1$. Also show that the above power series represents a continuous function S(z) at each point inside the circle of convergence $|z - z_0| = R$.

(b) Answer any **two** of the following briefly : **4**

(i) Find the value of the integral $\displaystyle\int_{|z - i| = 1} \frac{dz}{(z - i)^{n+3}}$

(ii) Specifying the domains, give two Laurent series expansions in powers of z for the function $f(z) = \dfrac{1}{z^2 (1 - z)}$.

(iii) Represent the function $f(z) = \dfrac{z + 1}{z - 1}$ by a Maclaurin Series and a Laurent Series specifying the domains.

(c) Answer all of the following very briefly : **3**

(i) Find Laurent Series for $f(z) = \dfrac{1}{4z - z^2}$ which is valid in $0 < | z | < 4$.

(ii) From the Maclaurin Series expansion of cos z, find the same for cosh z.

(iii) If $f(z) = \sin z^2$, find $f^{(4n)}(0)$ and $f^{(2n+1)}(0)$ for the non-negative integers n.

2. (a) State and prove Cauchy's Residue Theorem. Also evaluate the integral **7**

$$\int_{|z| = 2} \frac{5z - 2}{z(z - 1)}\, dz$$

OR

Suppose f is analytic at z_0. Show that z_0 is a zero of order m of f if and only if there is a function g which is analytic and nonzero at z_0 such that $f(z) = (z - z_0)^m g(z)$. Find all zeros and also their orders for the function $f(z) = z(e^z - 1)(z - 1)^3$.

(b) Answer any **two** of the following briefly : **4**

(i) Find $\underset{z = i}{\text{Res}} \dfrac{\text{Log } z}{(z^2 + 1)^2}$

(ii) Find $\underset{z = -1}{\text{Res}} \dfrac{z^{1/4}}{(z + 1)}$

(iii) Find $\underset{z = -1}{\text{Res}} \; - i \dfrac{z}{(z^4 + 4)}$

(c) Answer all of the following very briefly : **3**

(i) When do you say that z_0 is an isolated singular point of function f ? Give an example of a function which has exactly five distinct isolated singular points.

(ii) What type of singularity does the function $f(z) = \dfrac{z - \sin z}{z^3}$ have at origin ? Justify.

(iii) What type of singularity does the function $f(z) = e^{\frac{1}{z}}$ have at origin ? Justify.

3. (a) Show that any entire, bounded function is constant. Also state and prove the Fundamental Theorem of Algebra. **7**

OR

Suppose f(z) is analytic and $|f(z)| \leq |f(z_0)|$ on $|z - z_0| < \varepsilon$. Show that f(z) has the constant value $f(z_0)$ throughout the neighbourhood.

(b) Answer any **two** of the following briefly : **4**

(i) Under what extra condition does the Minimum Modulus Principle (just like the Maximum Modulus Principle) hold ? Show by appropriate example that the Minimum Modulus Principle is false in general.

(ii) Suppose that f(z) is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound; that is $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

(iii) Let $f(z) = (z + i)^2$ and R be the closed triangular region determined by 0, 2 and i. Give geometric argument and determine the points on R where the maximum and minimum of $|f(z)|$ occurs.

(c) Answer all of the following very briefly : **3**

(i) Let R denote the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$. Find the point in R where $f(z) = |\sin z|$ has the maximum value.

(ii) What is Gauss's Mean Value Theorem ? Why is it called so ?

(iii) Suppose f is an entire function such that $|f(z)| \leq A\,|z|$ for all z, where A is a fixed positive constant. Show that $f(z) = a_1 z$ where a_1 is a complex constant.

4. (a) Using residues evaluate the improper integrals and show that : **7**

(i) $\int_{0}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{32}$

(ii) P.V. $\int_{-\infty}^{\infty} \frac{x \sin x dx}{x^2+2x+2} = \frac{\pi}{e}(\sin 1 + \cos 1)$

OR

(i) Suppose that we have under appropriate assumptions including $|f(z)| \leq M_R$ on the semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$ with $M_R \to 0$ as R ends to ∞. Using Jordan's inequality show that for $a > 0$

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

(ii) Show that $\int_{\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$

(b) Answer any **two** of the following briefly : **4**

(i) Show that $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2\pi}$

(ii) Show that $\int_{0}^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$

(iii) Show that $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

(c) Answer all of the following very briefly : **3**

(i) How do you define the Improper Integral $\int_{-\infty}^{\infty} f(x)\, dx$?

(ii) How do you define the Improper Integral P.V. $\int_{-\infty}^{\infty} f(x)\, dx$?

(iii) Show by example that the two definitions of Improper Integrals are in general not equivalent.

5. (a) Using the function $f(z) = \frac{(\log z)^2}{z^2 + 1} \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right)$.

Derive integration formulas : **7**

$$\int_0^\infty \frac{(ln\, x)^2}{x^2 + 1} \, dx = \frac{\pi^3}{8} \text{ and } \int_0^\infty \frac{ln\, x}{x^2 + 1} \, dx = 0$$

You may use the result $\int_0^\infty \frac{1}{x^2 + 1} \, dx = \frac{\pi}{2}$.

OR

Suppose f is meromorphic in the domain interior to a positively oriented simple closed contour C, and f is analytic and nonzero on C. Then show that the winding number of $\Gamma = f(C)$ around origin is given by

$$\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P$$

What are Z and P ?

(b) Answer any **two** of the following briefly : **4**

(i) Determine the number of zeros (counting multiplicities) of the polynomial $2z^4 - 2z^3 + 2z^2 - 2z + 9$ inside the circle $|z| = 1$.

(ii) Find the winding number of the image of the unit circle under the map $f(z) = \frac{(2z - 1)^7}{z^3}$.

(iii) Show that every linear fractional transformation, with one exception, has at most two fixed points in the extended complex plane. State clearly as to what is this exception ?

(c) Answer all of the following very briefly : **3**

(i) Give definition of the linear fractional transformation T from the extended complex plane $\mathbb{C} \cup \{\infty\}$ onto the extended complex plane $\mathbb{C} \cup \{\infty\}$.

(ii) "Every linear fractional transformation T has at least two fixed points in the extended complex plane." Is the statement true ? Justify your answer.

(iii) Find all the fixed points of $Tz = \frac{z - 1}{z + 1}$ in the extended complex plane.